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EFFECT OF STRESS-FREE EDGES IN PLANE SHEAR OF A RECTANGULAR ORT--ETC(U)
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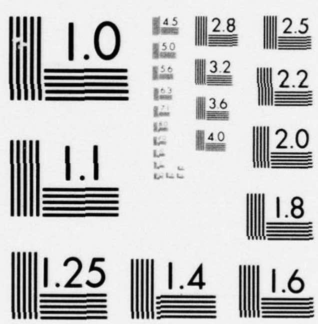
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UNIVERSITY OF CALIFORNIA, SAN DIEGO
Department of Applied Mechanics and Engineering Sciences
La Jolla, California 92093

EFFECT OF STRESS-FREE EDGES IN PLANE SHEAR OF A RECTANGULAR ORTHOTROPIC REGION

by S. Nair

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Effect of Stress-free Edges in Plane Shear of a Rectangular Orthotropic Region

S. Nair

University of California, San Diego
La Jolla, California

ABSTRACT

The plane elastic problem of a rectangular orthotropic region is considered; subject to the boundary conditions of prescribed equal and opposite tangential displacements and zero normal displacements on the upper and lower edges and zero stresses on the remaining edges. The effect of the stress-free edges on the stiffness coefficient relating the tangential displacement and the corresponding shearing force is estimated in the form of upper and lower bounds for this coefficient.

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INTRODUCTION

In what follows we consider the plane problem of a rectangular elastic region which represents an orthotropic lamina of unit thickness in plane stress or the cross section of an infinitely long orthotropic pad in plane strain. The boundary conditions consist of prescribed equal and opposite tangential and zero normal displacements on the upper and lower edges and zero tractions on the remaining two edges. Our objective here is to obtain upper and lower bounds for the stiffness coefficient relating the prescribed tangential displacement to the shear force required to produce this displacement. An elementary approximate solution to this problem may be obtained by considering a state of pure shear within the region, which would necessarily exclude the effect of the stress-free edges. In previous work [1], Read obtained bounds for the stiffness coefficient including the effect of the stress-free edges in the case of isotropic materials through the use of the Prager-Synge hypercircle method [2]. Our approach here consists in the simultaneous application of the principle of minimum potential energy and the principle of minimum complementary energy to bound the stiffness coefficient. These energy principles have been used previously to obtain bounds for influence coefficients associated with certain two dimensional boundary value problems in the case of cantilever beams [3] and in the case of cylindrical shells [4]. Our approach differs from that in [1] in the interpretation of the bounding functionals and generalizes the work in [1] by considering orthotropy and by removing a restriction used in [1], that the thickness to length ratio of the rectangular region is small compared to unity. The general bound results obtained for the orthotropic case are specialized to the isotropic case so as to obtain improved numerical results for the case considered in [1].

We complement our bound calculations by presenting exact expressions for the stiffness coefficient, through appropriate modification of previous results of Reissner [5] and Hildebrand [6], in the limiting-type orthotropic cases for which Young's modulus in the direction normal to the stressfree edges takes on the values zero or infinity, respectively.

THE BOUNDARY VALUE PROBLEM

We consider a rectangular region with boundaries $x = \pm a$ and $y = \pm c$. We assume that the boundary portions $y = \pm c$ are subjected to uniform displacements $\pm U$ in the x-direction and that displacements in the y-direction are prevented. The boundary portions $x = \pm a$ are assumed to be traction free.

We have as differential equations for stresses and strains

$$\sigma_{x,x} + \tau_{,y} = 0 \quad , \quad \tau_{,x} + \sigma_{y,y} = 0 \tag{1}$$

and

$$\epsilon_x = u_{,x} \quad , \quad \epsilon_y = v_{,y} \quad , \quad \gamma = u_{,y} + v_{,x} \tag{2}$$

and therewith as boundary conditions

$$y = \pm c \quad ; \quad u = \pm U \quad , \quad v = 0 \tag{3}$$

$$x = \pm a \quad ; \quad \sigma_x = 0 \quad , \quad \tau = 0 \tag{4}$$

We assume the medium to be orthotropic with constitutive relations

$$\epsilon_x = \frac{\sigma_x}{E_x} - \bar{\nu}\frac{\sigma_y}{E_m} \quad , \quad \epsilon_y = \frac{\sigma_y}{E_y} - \bar{\nu}\frac{\sigma_x}{E_m} \quad , \quad \gamma = \frac{\tau}{G} \tag{5}$$

where $E_m = \sqrt{E_x E_y}$, with E_x, E_y, $\bar{\nu}$ and G being constants.

For the case of isotropy and plane stress we have

$$E_x = E_y = E, \; \bar{\nu} = \nu \text{ and } G = \tfrac{1}{2} E/(1 + \nu) \tag{6a}$$

and for the case of isotropy and plane strain the corresponding relations are

$$E_x = E_y = E/(1 - \nu^2), \quad \bar{\nu} = \nu/(1 - \nu) \text{ and } G = \tfrac{1}{2} E/(1 + \nu). \tag{6b}$$

For the strain energy to be positive definite we must have $|\bar{\nu}| < 1$.

The prescribed edge displacements $u(x, \pm c) = \pm U$ are associated with forces

$$\pm P = \int_{-a}^{a} \tau(x, \pm c)dx, \tag{7}$$

in the form

$$P = KU, \tag{8}$$

with the values of the stiffness coefficient K being the principle objectives of the following considerations.

As an elementary approximation we may assume that the elastic region is in a state of pure shear, that is,

$$u = Uy/c, \quad \tau = U/Gc, \quad v = \sigma_x = \sigma_y = 0 \tag{9}$$

with the corresponding value of the stiffness coefficient being given by

$$K_o = 2Ga/c. \tag{9}$$

UPPER AND LOWER BOUND EXPRESSIONS FOR K

From the principles of minimum potential energy and minimum complementary energy we have as inequalities for the work quantity PU,

$$\tilde{I}_s \le PU \le \tilde{I}_d \tag{11}$$

where

$$\tilde{I}_s = 2U \int_{-a}^{a} \tilde{\tau}(x,c)dx - \tfrac{1}{2} \int_{-c}^{c} \int_{-a}^{a} \left\{ \frac{\tilde{\sigma}_x^2}{E_x} - \frac{2\bar{\nu}\tilde{\sigma}_x\tilde{\sigma}_y}{E_m} + \frac{\tilde{\sigma}_y^2}{E_y} + \frac{\tilde{\tau}^2}{G} \right\} dxdy \tag{12}$$

and

$$\tilde{I}_d = \frac{1}{2} \int_{-c}^{c} \int_{-a}^{a} \left\{ E_x^* \tilde{\epsilon}_x^2 + 2\bar{\nu} E_m^* \tilde{\epsilon}_x \tilde{\epsilon}_y + E_y^* \tilde{\epsilon}_y^2 + G\tilde{\gamma}^2 \right\} dxdy \tag{13}$$

where

$$(E_x^*, E_m^*, E_y^*) = (E_x, E_m, E_y)/(1-\bar{\nu}^2). \tag{14}$$

For (11) to be valid the stress components $\tilde{\sigma}$, $\tilde{\tau}$ must satisfy the equilibrium equations (1) and the stress boundary conditions (4) and the strain components $\tilde{\epsilon}$, $\tilde{\gamma}$ must be derived from displacements $\tilde{u}$, $\tilde{v}$ which satisfy the boundary conditions (3).

Substituting for P from (8), equation (11) becomes a system of inequalities for K,

$$\tilde{I}_s/U^2 \le K \le \tilde{I}_d/U^2 . \tag{15}$$

In order to obtain upper bounds K_u and lower bounds K_ℓ we consider the set of displacement and stress approximations

$$\tilde{u} = U\left[\frac{y}{c} + \left(\frac{y}{c} - \frac{y^3}{c^3}\right) F_2(\xi)\right] ; \quad \tilde{v} = \frac{U}{\rho}\left(1 - \frac{y^2}{c^2}\right) F_1(\xi) \tag{16}$$

$$\tilde{\sigma}_x = \frac{2GU}{\rho c}\frac{y}{c} f_2(\xi) , \quad \tilde{\tau} = \frac{GU}{c}\left[f_1(\xi) + \left(\frac{1}{5} - \frac{y^2}{c^2}\right) f_2'(\xi)\right]$$

$$\tilde{\sigma}_y = \frac{GU\rho}{c}\left[\left(\frac{1}{3}\frac{y^3}{c^3} - \frac{1}{5}\frac{y}{c}\right) f_2''(\xi) - \frac{y}{c} f_1'(\xi)\right] \tag{17}$$

where

$$\xi = x/a \quad \text{and} \quad \rho = c/a \tag{18}$$

and where primes indicate differentiation with respect to ξ.

The expressions in (16) satisfy the displacement boundary conditions for all choices of F_1 and F_2. The expressions in (20) satisfy the equilibrium equations (1) for all choices of f_1 and f_2. In order to satisfy the stress boundary conditions (4) they must be such that

$$f_1(\pm 1) = f_2(\pm 1) = f_2'(\pm 1) = 0 \; . \tag{19}$$

We note that the approximating functions used in [1] are subsets of the above class of functions.

Obvious symmetries pertaining to the boundary value problem as stated require that F_2 and f_1 must be even functions of ξ whereas F_1 and f_2 must be odd functions of ξ.

Using (16),(13) and (17) in (12) we obtain, after completing all y-integrations,

$$\frac{K_u}{K_o} = \frac{1}{2}\int_{-1}^{1}\Big\{1 + \frac{4}{3}F_1' + \frac{4}{5}F_2^2 + \frac{8E_x^*\rho^2}{105G}(F_2')^2 + \frac{8}{15}F_2F_1'$$

$$- \frac{8}{15}\frac{\bar{\nu}E_m^*}{G}F_2'F_1 + \frac{4E_y^*}{3G\rho^2}F_1^2 + \frac{8}{15}(F_1')^2\Big\}\,d\xi. \tag{20}$$

$$\frac{K_\ell}{K_o} = \frac{1}{2}\int_{-1}^{1}\Big\{2f_1 - \frac{G\rho^2}{3E_y}(f_1')^2 - f_1^2 + \frac{4}{15}\left(1 + 5\frac{\bar{\nu}G}{E_m}\right)f_1 f_2'$$

$$- \frac{4}{3}\frac{G}{E_x\rho^2}f_2^2 - \frac{8}{75}(f_2')^2 - \frac{4}{1575}\frac{G\rho^2}{E_y}(f_2'')^2\Big\}\,d\xi. \tag{21}$$

where $K_o = 2Ga/c$. It remains to minimize the functional (20) and to maximize the functional (21). We note that our elementary approximation (9) corresponds to the assumption $F_1 = F_2 = 0$ in (16). Furthermore, from (20) we have $K_u \leq K_o$.

FIRST APPROXIMATION UPPER AND LOWER BOUNDS

We obtain a relatively simple formula for the upper bound by setting $F_2(\xi) = 0$ in the expressions (16) for $\tilde{u}$ and $\tilde{v}$. Then the upper bound coefficient (20) may be written as

$$\frac{K_u}{K_o} = \frac{1}{2} \int_{-1}^{1} \left\{ 1 + \frac{4}{3} F_1' + \frac{4E_y^*}{3G\rho^2} F_1^2 + \frac{8}{15} (F_1')^2 \right\} d\xi \ . \tag{22}$$

Minimization of (22) results in the Euler differential equation

$$F_1'' - \frac{5E_y^*}{2G\rho^2} F_1 = 0 \tag{23}$$

and the Euler boundary conditions

$$F_1'(\pm 1) = -\frac{5}{4} \tag{24}$$

where, as stated earlier, $F_1(\xi)$ is an odd function of ξ.

The minimum value of K_u follows from (22) and (23) as

$$\frac{K_{u1}}{K_o} = 1 + \frac{2}{3} F_1(1) \ . \tag{25}$$

The solution of (25) subject to the boundary conditions (24) is,

$$F_1 = -\frac{5\rho}{4\Lambda_o} \frac{\sinh \Lambda_o \xi/\rho}{\cosh \Lambda_o/\rho} \tag{26}$$

where

$$\Lambda_o = \sqrt{5E_y^*/2G} \tag{27}$$

Using (26) in (25) we obtain the first approximation upper bound,

$$\frac{K_{u1}}{K_o} = 1 - \frac{5}{6} \frac{\rho}{\Lambda_o} \tanh \frac{\Lambda_o}{\rho} \tag{28}$$

We next calculate a lower bound by setting $f_2 = 0$ in equation (17). The lower bound coefficient (21) when $f_2 = 0$ reduces to,

$$\frac{K_{\ell 1}}{K_o} = \frac{1}{2}\int_{-1}^{1} \left\{2f_1 - \frac{G\rho^2}{3E_y}(f_1')^2 - f_1^2\right\} d\xi \tag{29}$$

with the constraint conditions $f_1(\pm 1) = 0$. Maximization of (29) results in the Euler differential equation

$$f_1'' - \frac{3E_y}{G\rho^2} f_1 = -\frac{3E_y}{G\rho^2} \tag{30}$$

and in a maximum value of $K_{\ell 1}$ in the form

$$\frac{K_{\ell 1}}{K_o} = \frac{1}{2}\int_{-1}^{1} f_1 \, d\xi \, . \tag{31}$$

Use of the solution

$$f_1 = 1 - \frac{\cosh \lambda_o \xi/\rho}{\cosh \lambda_o/\rho} \tag{32}$$

where

$$\lambda_o = \sqrt{3E_y/G} \tag{33}$$

gives the lower bound expression

$$\frac{K_{\ell 1}}{K_o} = 1 - \frac{\rho}{\lambda_o} \tanh \frac{\lambda_o}{\rho} \, . \tag{34}$$

Having the first approximation bound formulas (28) and (34) we note that we may approximate the hyperbolic tangent functions by unity for sufficiently large values λ_o and Λ_o, say for λ_o, $\Lambda_o > 4\rho$, to obtain

$$1 - \sqrt{\frac{G}{3E_y}}\,\rho \le \frac{K}{K_o} \le 1 - \sqrt{\frac{5G(1-\bar{\nu}^2)}{18E_y}}\,\rho \, . \tag{35}$$

Using (6a) and (6b) we obtain in the case of isotropic plane stress the bound inequality,

$$1 - \frac{1}{\sqrt{6(1+\nu)}}\,\rho \leqslant \frac{K}{K_o} \leqslant 1 - \frac{\sqrt{5}}{6}\sqrt{1-\nu}\;\rho \tag{36a}$$

and in the case of plane strain the inequality,

$$1 - \sqrt{\frac{1-\nu}{6}}\,\rho \leqslant \frac{K}{K_o} \leqslant 1 - \frac{\sqrt{5}}{6}\sqrt{\frac{1-2\nu}{1-\nu}}\;\rho \tag{36b}$$

The bounds given by (36b) are identical to those obtained in [1]. As important properties of the bound formulas (28) and (34) we note that a) the effect of the stress free edges is to decrease the stiffness and b) these first approximation bound coefficients are independent of the elastic constant E_x.

IMPROVED UPPER AND LOWER BOUNDS

In order to improve the bounds (28) and (34) we next consider the complete expressions (20) and (21) for K_u and K_ℓ. Minimization of (20) with respect to F_1 and F_2 gives the Euler differential equations

$$F_1'' - \frac{5}{2}\frac{E_y^*}{G\rho^2}F_1 + \frac{1}{2}\left(1 + \frac{\bar{\nu}E_m^*}{G}\right)F_2' = 0 \tag{37a}$$

$$F_2'' - \frac{21}{2}\frac{G}{E_x^*\rho^2}F_2 - \frac{7}{2}\frac{G}{E_x^*\rho^2}\left(1 + \frac{\bar{\nu}E_m^*}{G}\right)F_1' = 0 \tag{37b}$$

and the Euler boundary conditions

$$F_1'(1) + \frac{1}{2}F_2(1) = -\frac{5}{4} \quad , \quad F_2'(1) - \frac{7}{2}\frac{\bar{\nu}E_m^*}{E_x^*\rho^2}F_1(1) = 0 \tag{38}$$

Appropriate integrations by parts and use of (37) and (38) enables us to obtain the minimum value of (20) in the form,

$$\frac{K_{u2}}{K_o} = 1 + \frac{2}{3} F_1(1) \tag{39}$$

Upon solving the system of differential equations (37) subject to the boundary conditions (38) and substituting for $F_1(1)$ in (39) we obtain

$$\frac{K_{u2}}{K_o} = 1 - \frac{5\rho}{6} \frac{\Lambda_2^2 - \Lambda_1^2}{\chi} \tanh \frac{\Lambda_1}{\rho} \tanh \frac{\Lambda_2}{\rho} \tag{40}$$

where Λ_i are the roots with positive real parts of the equation

$$\Lambda^4 - \left[(1 - \frac{7\bar{\nu}^2}{10}) \frac{5E^*_y}{2G} + \frac{35G}{4E^*_x} - \frac{7}{2} \frac{\bar{\nu}E^*_m}{E^*_x}\right] \Lambda^2 + \frac{105}{4} \frac{E^*_y}{E^*_x} = 0 \tag{41}$$

and where

$$\chi = \frac{G}{G + \bar{\nu}E^*_m} \left[\chi_1 (\Lambda_2 \; tanh \frac{\Lambda_2}{\rho} - \Lambda_1 \; tanh \frac{\Lambda_1}{\rho})\right.$$

$$\left. + \chi_2(\Lambda_2^3 \tanh \frac{\Lambda_2}{\rho} - \Lambda_1^3 \tanh \frac{\Lambda_1}{\rho}) + \chi_3(\Lambda_2 \tanh \frac{\Lambda_1}{\rho} - \Lambda_1 \tanh \frac{\Lambda_2}{\rho})\right] \tag{42}$$

with

$$\chi_1 = \frac{7\bar{\nu}E^*_m}{6G} \Lambda_1 \Lambda_2 - \left(\frac{5E^*_y}{2G}\right)^2 \frac{1 - .7\bar{\nu}^2}{\Lambda_1 \Lambda_2} \quad , \quad \chi_2 = \frac{5E^*_y}{2G} \frac{1}{\Lambda_1 \Lambda_2}$$

$$\chi_3 = \frac{\bar{\nu}E^*_m}{G} \left[\frac{5E^*_y}{2G} (1 - .7\bar{\nu}^2) - \frac{7\bar{\nu}E^*_m}{4E^*_x}\right] . \tag{43}$$

When $\Lambda_i/\rho \geq 4$, K_{u2} may be approximated by its asymptotic form,

$$\frac{K_{u2}}{K_o} \approx 1 - \rho\;(\Lambda_1 + \Lambda_2) \Big/ \left[\frac{3E_y}{G} \frac{1 - .7\bar{\nu}^2}{1 - \bar{\nu}^2} + \Lambda_1 \Lambda_2\right] . \tag{44}$$

Next we proceed to obtain, in a similar fashion, an improved lower bound for K. However, maximization of the functional (21) as it stands would require the solution of a sixth order differential equation. In order to avoid increasing the order of our problem beyond the fourth we shall in what follows, maximize the expression (21) by assuming $f_1(\xi)$ as given by the first approximation calculations. In this way the resulting Euler differential equation is the fourth order equation,

$$f_2'''' - \frac{42E_y}{G\rho^2} f_2'' + \frac{525E_y}{E_x\rho^4} f_2 = -\frac{105E_y}{2G\rho^2}\left(1 + 5\frac{\bar{\nu}G}{E_m}\right)f_1' \quad , \tag{45}$$

with f_1' corresponding to f_1 in (32).

The constraint conditions on f_2 in (19) and the differential equation (45) in conjunction with appropriate integrations by parts enable us to write the maximum value of the functional (21) in the form

$$\frac{K_{\ell 2}}{K_o} = 1 - \frac{\rho}{\lambda_o}\tanh\frac{\lambda_o}{\rho} - \frac{1}{15}\left(1 + 5\frac{\bar{\nu}G}{E_m}\right)\int_{-1}^{1} f_1' f_2 d\xi \quad . \tag{46}$$

Upon solving the non-homogeneous equation (45) subject to the appropriate boundary conditions and evaluating the integral in (46), we obtain as improved lower bound expression

$$\frac{K_{\ell 2}}{K_o} = 1 - \frac{\rho}{\lambda_o} + \frac{7\rho}{6}\,\frac{\lambda_o^4 \varkappa(1 + 5\bar{\nu}G/E_m)}{\lambda_o^4 - (\lambda_1^2 + \lambda_2^2)\lambda_o^2 + \lambda_1^2\lambda_2^2} \quad , \tag{47}$$

where λ_i are the roots with positive real parts of the equation

$$\lambda^4 - 42\frac{E_y}{G}\lambda^2 + 525\frac{E_y}{E_x} = 0 \quad , \tag{48}$$

and where

$$\varkappa = \varkappa_o + \varkappa_1 \quad , \tag{49}$$

with

$$\varkappa_o = \frac{\tanh \lambda_o/\rho}{\lambda_o} - \frac{1}{\rho \cosh^2 \lambda_o/\rho} \quad , \tag{50}$$

$$\varkappa_1 = \frac{2(\lambda_2^2 - \lambda_1^2)}{(\lambda_o^2 - \lambda_2^2)(\lambda_o^2 - \lambda_1^2)}$$

$$\times \frac{(\lambda_1 \tanh \lambda_o/\rho - \lambda_o \tanh \lambda_1/\rho)(\lambda_2 \tanh \lambda_o/\rho - \lambda_o \tanh \lambda_2/\rho)}{\lambda_1 \tanh \lambda_2/\rho - \lambda_2 \tanh \lambda_1/\rho} \quad . \tag{51}$$

When $\lambda_i/\rho \geq 4$, $K_{\ell 2}$ may be approximated by its asymptotic form,

$$\frac{K_{\ell 2}}{K_o} = 1 - \frac{\rho}{\lambda_o}\left[1 - \frac{7}{6}\,\frac{\lambda_o^4(1 + 5\bar{\nu}G/E_m)^2}{(\lambda_o + \lambda_1)^2(\lambda_o + \lambda_2)^2}\right] \quad . \tag{52}$$

We note that the asymptotic formulas (35) , (44), and (52) correspond to the assumption of the existence of boundary layers in the neighborhood of the edges $x = \pm a$. In the expressions for K_u and K_ℓ the leading terms represent the interior solution contribution in the stiffness coefficient corresponding to a state of pure shear. The remaining terms represent the boundary layer contributions. Within these boundary layers the interior shear stress U/Gc undergoes a rapid transition to attain the value zero at $x = \pm a$. The approximate boundary layer thickness obtained in [1] for the isotropic plane strain case may be generalized to the case of orthotropy by observing our first approximation solutions. For the orthotropic case we have from equations (26), (27), (32) and (33) as a condition for the existence of boundary layers

$$\rho \leq \sqrt{E_y/G} \quad , \tag{53}$$

and the corresponding width b_o of these boundary layers is given by

$$b_o/c = O\,(\sqrt{G/E_y}) \; . \tag{54}$$

Our improved bound calculations show that there exists an additional set of boundary layers adjacent to the edges $x = \pm a$. We demonstrate this by considering the characteristic equation (48) corresponding to the lower bound calculation which may be considered more relevent in the context of a boundary layer estimate pertaining to stress transitions. As conditions for the existence of these additional boundary layers we have

$$E_m/G = O(1) \quad , \qquad \rho \leq \sqrt[4]{E_y/E_x} \tag{55}$$

$$1 \ll E_m/G \quad , \qquad \rho \leq \sqrt{E_y/G} \tag{56}$$

The widths of these boundary layers are given by

$$E_m/G = O(1) \quad , \qquad b_i/c = O\left(\sqrt[4]{E_x/E_y}\right) \quad , \qquad i = 1, 2 \tag{57}$$

$$1 \ll E_m/G \quad , \qquad b_1/c = O\left(\sqrt{E_x/G}\right) \quad , \qquad b_2/c = O\left(\sqrt{G/E_y}\right) \tag{58}$$

It is apparent from (54), (57), and (58) that there exist two boundary layers near each stress-free edge, for all values of the parameter E_m/G.

As far as the numerical calculation of the bounds is concerned we may use the asymptotic formulas (35), (44) and (52) whenever,

$$\text{Min}\,\{\text{Re}[\Lambda_i, \lambda_i], \Lambda_o, \lambda_o\} \geq 4\rho \quad , \tag{59}$$

as these formulas involve the approximation $\tanh(\Lambda, \lambda)/\rho \approx 1$.

The roots of the characteristic equations (41) and (48) are in general complex. As conditions for real roots we must have

$$\frac{G}{E_m} \leq \frac{2}{\sqrt{105}} \left| \frac{5}{2} \frac{1 - .7\bar{\nu}^2}{1 - \bar{\nu}^2} + 35 \frac{1 - \bar{\nu}^2}{4} \frac{G^2}{E_m^2} - \frac{7\bar{\nu}}{2} \frac{G}{E_m} \right| \tag{60}$$

for real Λ_1 and Λ_2 and

$$\frac{G}{E_m} \leq 1 \tag{61}$$

for real λ_1 and λ_2 .

The improved upper (or lower) bound obtained here is not valid when the equality sign holds in (60) (or in (61)). However, expressions for the bound coefficients may be deduced from the formula (40) (or (47)) by taking the limit as $\Lambda_1 \to \Lambda_2$ (or $\lambda_1 \to \lambda_2$).

Isotropic plane stress case. In the case of isotropic plane stress we use (6a) in (40) and (47) to evaluate the improved bounds. Fig. 1 shows a plot of K_{u1}, K_{u2}, $K_{\ell 2}$ and $K_{\ell 1}$ against $\rho = c/a$ for the plane stress problem when $\nu = 1/3$. The plots show that the exact formulas and the corresponding asymptotic formulas (linear functions of c/a) are indistinguishable in the range $0 \le c/a \le 1$, with $c/a = 1$ representing a square lamina. The maximum error in employing the average of the improved bounds in place of the exact stiffness coefficient comes out to be about 0.9% when $c/a = 1$ and this error is linearly decreasing with decreasing values of c/a. The corresponding error in using the first approximation bound expressions comes out to be about 4%.

Orthotropic case. In the case of orthotropic materials the bound formulas (40) and (47) suggest the use of the parameters

$$\mu = G/\sqrt{E_x E_y} \quad , \qquad \rho^* = \sqrt{G/E_y} \;\; c/a \tag{62}$$

with μ representing the effect of the elastic coefficient E_x. Fig. 2 shows plots of K_{u1}, K_{u2}, $K_{\ell 2}$ and $K_{\ell 1}$ against ρ^* when $\bar{\nu} = 1/3$, $\mu = 10$. As ρ^* increases the difference between the upper bound K_{u1} and the exact result becomes significant, although the remaining bounds K_{u2}, $K_{\ell 2}$ and $K_{\ell 1}$ are extremely close. We note that for large values of ρ^*, if G and E_y are of the same order, $c/a \gg 1$. Then our problem is equivalent to that of a vertical beam with length 2c and thickness 2a subjected to prescribed displacements (zero displacement in the axial and 2U in the tangential directions at one end and the other end fixed). We have from (34)

$$K_{\ell 1} = \frac{2Ga}{c}\left[1 - \frac{c}{a}\sqrt{\frac{G}{3E_y}}\tanh\sqrt{\frac{3E_y}{G}}\frac{a}{c}\right] \tag{63}$$

and when $c/a >> 1$, using the series expansion of $\tanh \lambda$ for $\lambda << 1$, we get

$$K_{\ell 1} = 2E_y\left(\frac{a}{c}\right)^3\left[1 - \frac{6}{5}\frac{E_y}{G}\frac{a^2}{c^2} + 0\left(\frac{a^4}{c^4}\right)\right] \tag{64}$$

which is the beam stiffness coefficient with a shear deformation factor 6/5.

EXACT VALUES OF STIFFNESS COEFFICIENTS FOR TWO LIMITING-TYPE ORTHOTROPY CASES

Exact expressions for the stiffness coefficients may be obtained in the two cases of limiting-type orthotropy by following the formulation in [5] in the case of $E_x = 0$ and by following the formulation in [6] in the case of $E_x = \infty$. These calculations are presented in an Appendix. Our final results show that

$$K_{E_x = 0} = K_o\left[1 - \frac{\rho^*}{\alpha}\tanh\frac{\alpha}{\rho^*}\right] \tag{65}$$

$$K_{E_x = \infty} = K_o\left[1 + \rho^* \sum_{n=0}^{\infty}\frac{1}{\alpha_n^3}\,\frac{\tanh \alpha_n/\rho^*}{1 - \rho^*(\tanh \alpha_n/\rho^*)/\alpha_n}\right]^{-1} \tag{66}$$

where

$$\alpha = \sqrt{3/(1 - \nu^2)}\ , \qquad \alpha_n = (n + \frac{1}{2})\,\pi \tag{67}$$

Fig. 3 shows plots of (65) and (66) for a wide range of ρ^*. These plots show that the effect of E_x on the stiffness is small when $\rho^* << 1$ in the entire range of the parameter $\mu \equiv G/\sqrt{E_x E_y}$.

In Fig. 3 we have also plotted the mean value of the closest bounds when $\mu = 10$. For the range of ρ^* shown in Fig. 2 the mean value of K_{u2} and $K_{\ell 2}$ differs from the exact values by less than 3.5% with the maximum of this error corresponding to $\rho^* \simeq 4$. The plot in Fig. 3 shows that the exact result for $E_x = 0$ $(\mu = \infty)$ may be used as an approximation for K when $\mu < 10$, $\rho^* < 1$, as in this range the curves for $\mu = 10$ and for $\mu = 0$ are not distinguishable.

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APPENDIX

a. <u>The case $E_x = 0$.</u> We write the resulting stress strain relations in the form

$$\sigma_x = 0, \; \sigma_y = E_y \epsilon_y / (1 - \bar{\nu}^2), \; \tau = G\gamma . \tag{A1}$$

Proceeding as in [5] the equilibrium equations (1) are satisfied by taking

$$\tau = f(x) \, , \quad \sigma_y = -yf'(x) \tag{A2}$$

The second equations in (A1) and (A2) along with the defining relation $\epsilon_y = v_{,y}$ then give as expression for v

$$v = \frac{1 - \bar{\nu}^2}{E_y} \left[-\frac{1}{2} y^2 f' + y\,g(x) + h(x) \right] \tag{A3}$$

Satisfaction of the boundary conditions $v(x, \pm c) = 0$ gives

$$g = 0 \, , \quad h = \frac{1}{2} c^2 f' \tag{A4}$$

and therewith

$$v = \frac{1 - \bar{\nu}^2}{E_y} \frac{c^2 - y^2}{2} f' . \tag{A5}$$

The constitutive relation for τ in (A1) in conjunction with (A5) and the defining relation $\gamma = u_{,y} + v_{,x}$ then gives further

$$u = \frac{yf}{G} - \frac{1 - \bar{\nu}^2}{E_y} \left(\frac{c^2 y}{2} - \frac{y^3}{6} \right) f'' \tag{A6}$$

upon taking account that u is an odd function of y.

We next obtain the differential equation for f by setting $u(x, c) = U$,

$$f'' - \frac{3}{1 - \bar{\nu}^2} \frac{E_y}{Gc^2} f = - \frac{3}{1 - \bar{\nu}^2} \frac{E_y}{c^3} U . \tag{A7}$$

The remaining boundary conditions $\tau(\pm a, y) = 0$ determine the solution of (A7) in the form,

$$f = \frac{GU}{c}\left[1 - \frac{\cosh \sqrt{3E_y/G(1-\bar{\nu}^2)}\; x/c}{\cosh \sqrt{3E_y/G(1-\bar{\nu}^2)}\; a/c}\right] . \tag{A8}$$

Use of (A8) in the expression for the stiffness coefficient,

$$K = \int_{-a}^{a} \tau(x, c)ds/U \tag{A9}$$

gives

$$\left(\frac{K}{K_o}\right)_{E_x = 0} = 1 - \sqrt{\frac{G(1-\bar{\nu}^2)}{3E_y}}\; \frac{c}{a} \tanh \sqrt{\frac{3E_y}{G(1-\bar{\nu}^2)}}\; \frac{a}{c} \tag{A10}$$

b. The case $E_x = \infty$. In this case the constitutive relations (5) reduce to

$$\epsilon_x = 0 \quad , \quad \epsilon_y = \sigma_y/E_y \quad , \quad \gamma = \tau/G . \tag{A11}$$

We consider $u(y)$ and $v(x, y)$ of the form

$$u = \frac{Py}{2Ga} + \sum_{n=0}^{\infty} u_n(y) \quad , \quad v = \sum_{n=0}^{\infty} X_n(x)Y_n(y) \quad , \tag{A12}$$

where P is a constant to be determined and, in order to satisfy the conditions $v(x, \pm c) = 0$,

$$Y_n(\pm c) = 0 \tag{A13}$$

Equation (A12) in conjunction with the constitutive relation for τ in (A11) gives

$$\tau = \frac{P}{2a} + G \sum_{n=0}^{\infty} [u_n^{\boldsymbol{\cdot}} + X_n' Y_n] \quad , \tag{A14}$$

where primes and dots denote differentiation with respect to the respective arguments.

From (1) we have as an expression for σ_x,

$$\sigma_x = -\int_{-a}^{x} \tau,_y dx \ . \tag{A15}$$

Use of (A14) in (A15) and satisfaction of the boundary conditions $\sigma_x(\pm a, y) = 0$, give

$$u_n^{\cdot\cdot} = [X_n(-a) - X_n(a)]Y_n^{\cdot}/2a \ . \tag{A16}$$

As v is odd in x and even in y, integration of (A16) gives

$$u_n^{\cdot} = -X_n(a)Y_n/a \ . \tag{A17}$$

From the second equations in (A11) and (A12) we have

$$\sigma_y = E_y \sum_{n=0}^{\infty} X_n Y_n^{\cdot} \ . \tag{A18}$$

Substituting for σ_y from (A18) and for τ from (A14) with u_n' as in (A17) into the equilibrium relation between σ_y and τ gives

$$\sum_{n=0}^{\infty} [E_y X_n Y_n^{\cdot\cdot} + G X_n'' Y_n] = 0 \ . \tag{A19}$$

The solutions of (A19) are taken in the form

$$Y_n = \cos \alpha_n y/c \quad , \quad X_n = A_n \sinh \sqrt{E_y/G}\, \alpha_n x/c \tag{A20}$$

where, in order to satisfy (A13),

$$\alpha_n = \frac{2n+1}{2} \pi \quad , \tag{A21}$$

and where the constants A_n remain to be determined.

We next use (A14) to write for the boundary values of τ

$$\tau(a, y) = \frac{P}{2a} + \frac{G}{c} \sum_{n=0}^{\infty} A_n \left[\sqrt{\frac{E_y}{G}} \alpha_n \cosh \sqrt{\frac{E_y}{G}} \alpha_n \frac{a}{c} - \frac{c}{a} \sinh \sqrt{\frac{E_y}{G}} \alpha_n \frac{a}{c} \right] \cos \alpha_n \frac{y}{c}$$

(A22)

A formal expansion of $P/2a$ in a Fourier series using the orthogonal eigenfunctions $\cos\alpha_n y/c$ and setting $\tau(a, y) = 0$ then gives

$$A_n = P \frac{\sin\alpha_n}{G\alpha_n} \left[\sinh \sqrt{\frac{E_y}{G}} \alpha_n \frac{a}{c} - \sqrt{\frac{E_y}{G}} \alpha_n \frac{a}{c} \cosh \sqrt{\frac{E_y}{G}} \alpha_n \frac{a}{c} \right]^{-1} \qquad (A23)$$

From (A14) and (A17), with $Y_n(c) = 0$, we see that

$$P = \int_{-a}^{a} \tau(x, c) dx \; . \qquad (A24)$$

Integration of (A17) with Y_n as in (A20) and substitution of the resulting u_n in (A12) finally gives

$$U = \frac{Pc}{2Ga} \left[1 + \frac{2c}{a} \sqrt{\frac{G}{E_y}} \sum_{n=0}^{\infty} \frac{1}{\alpha_n^3} \frac{\tanh \sqrt{E_y/G}\,\alpha_n a/c}{1 - (\tanh \sqrt{E_y/G}\,\alpha_n a/c)/\sqrt{E_y/G}\,\alpha_n a/c} \right] \qquad (A25)$$

The stiffness coefficient $K_{E_x = \infty}$ follows from (A25) as in (66).

We note that the above calculation is similar to that in [6] so far as the solution procedure is concerned. However, our boundary conditions are different from those considered in [6].

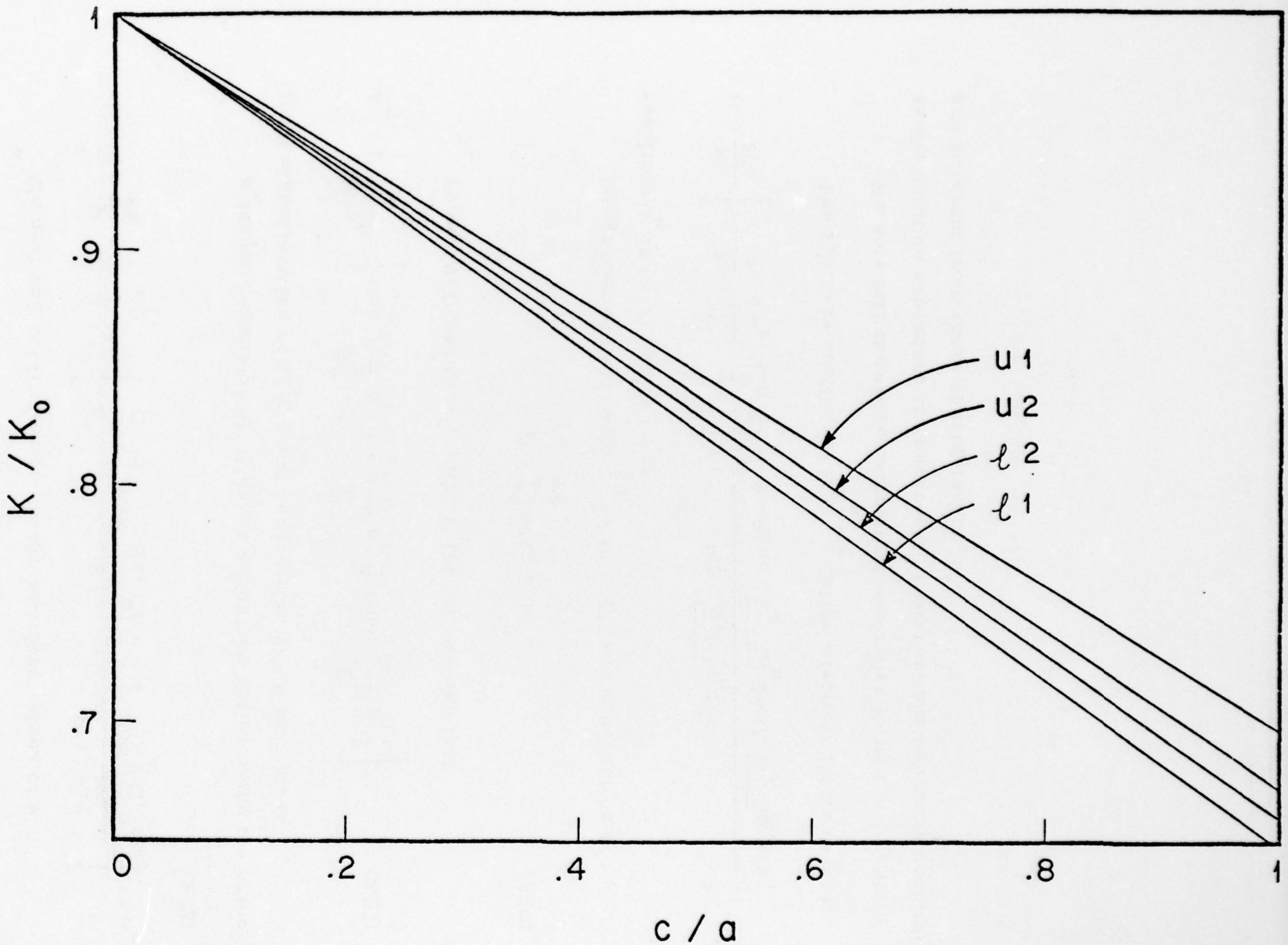

Figure 1. Values of stiffness coefficient bounds for isotropic materials with $\nu = 1/3$.

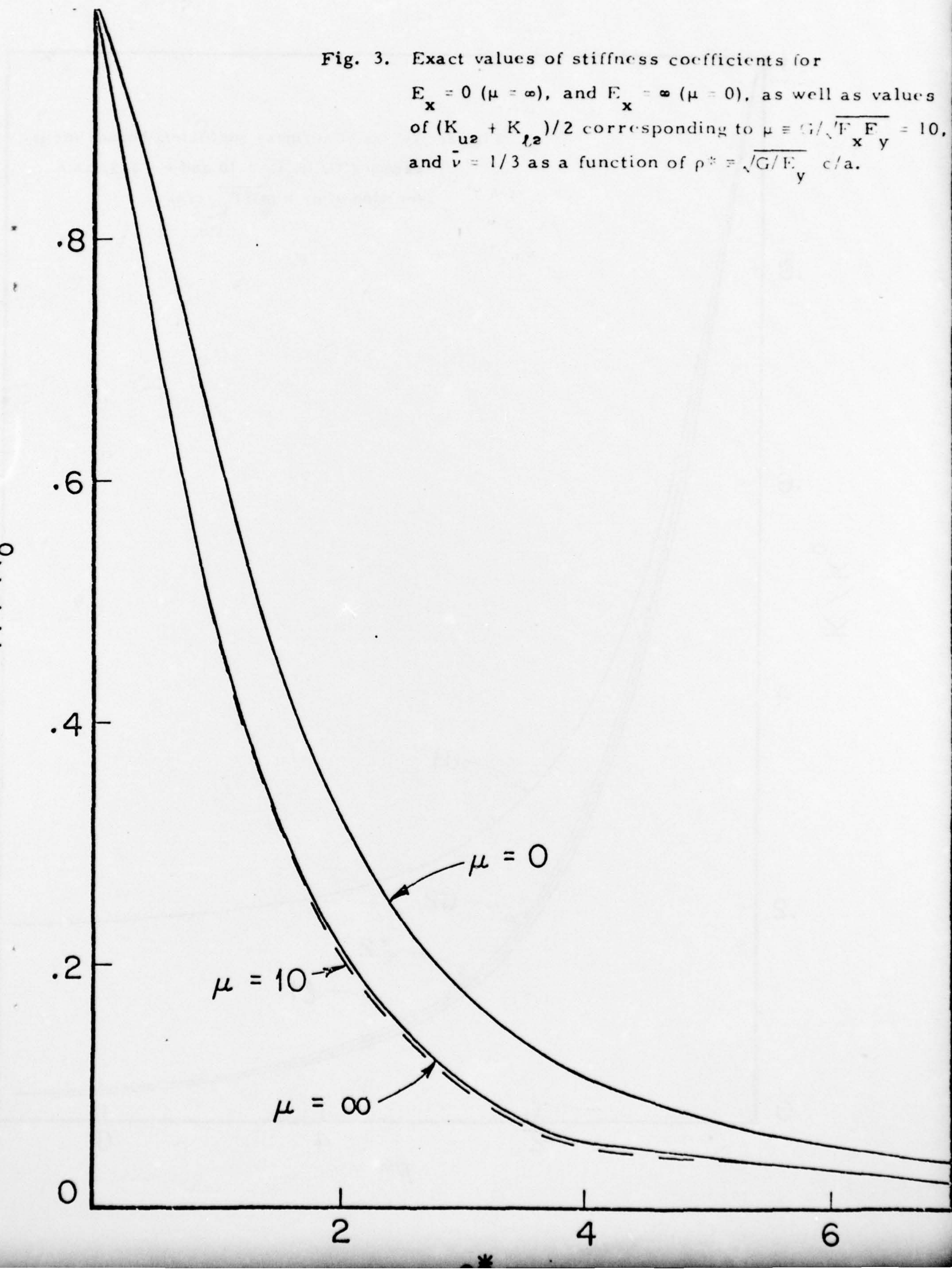

Fig. 3. Exact values of stiffness coefficients for $E_x = 0$ ($\mu = \infty$), and $E_x = \infty$ ($\mu = 0$), as well as values of $(K_{u2} + K_{\ell 2})/2$ corresponding to $\mu = G/\sqrt{E_x E_y} = 10$, and $\bar{\nu} = 1/3$ as a function of $\rho^* = \sqrt{G/E_y}\; c/a$.

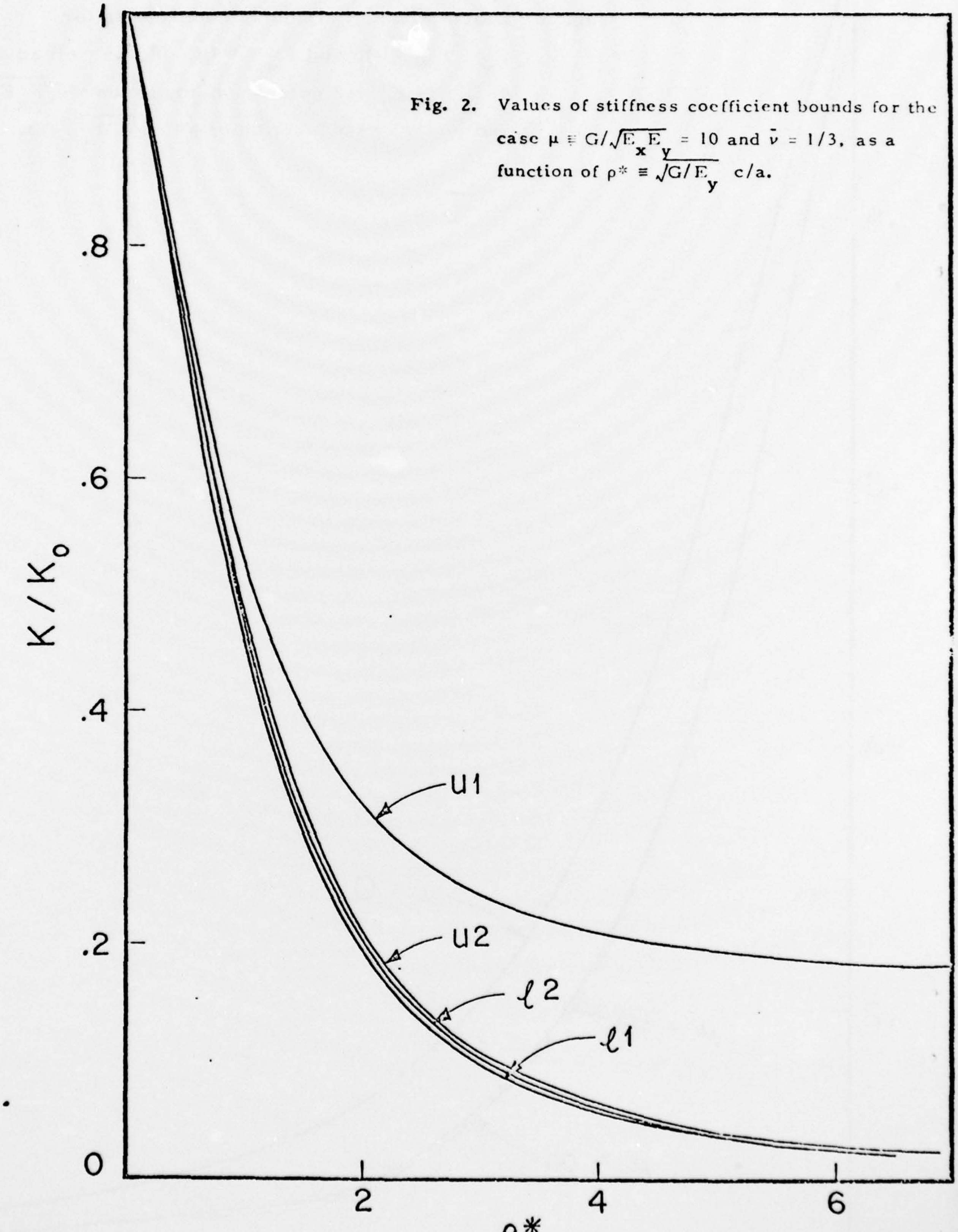

Fig. 2. Values of stiffness coefficient bounds for the case $\mu \equiv G/\sqrt{E_x E_y} = 10$ and $\bar{\nu} = 1/3$, as a function of $\rho^* \equiv \sqrt{G/E_y}\; c/a$.

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The plane elastic problem of a rectangular orthotropic region is considered; subject to the boundary conditions of prescribed equal and opposite tangential displacements and zero normal displacements on the upper and lower edges and zero stresses on the remaining edges. The effect of the stress-free edges on the stiffness coefficient relating the tangential displacement and the corresponding shearing force is estimated in the form of upper and lower bounds for this coefficient.

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